Rayleigh Flow ①

* Compressible flow with heating / cooling also called simple heat transfer flow'

* A frictionless flow process in a constant area duct with heat transfer is called Rayleigh flow.

* Change in flow parameters occurs only due to heating or cooling. Heating / cooling causes change in stagnation Temperature – Simple T_0-change flow.

* Application – Combustion chambers, intercoolers, regenerators

Assumptions

1) Constant area duct.
2) Frictional effects are small compare to heat transfer effect.
3) No mass addition or mass removal.
4) Perfect gas
5) Composition of gas doesnt change during flow.
6) Steady & 1-D flow
7) No external work
8) No body forces.

Governing Equations

1) Continuity Equation

$$\dot{m} = \rho_1 A C_1 = \rho_2 A C_2 = \text{Constant}$$

$$\frac{\dot{m}}{A} = \rho_1 C_1 = \rho_2 C_2 = G = \text{Constant}.$$

2) Momentum Equation (2)

T = Constant.

$$P_1A + \rho_1 A C_1^2 = P_2A + \rho_2 A C_2^2$$

3) Energy Equation.

Here heat addition / heat removal takes place hence h_0 value changes according.

$$\dot{q} = h_{02} - h_{01}$$

$$\dot{q} = \left(h_2 + \frac{C_2^2}{2}\right) - \left(h_1 + \frac{C_1^2}{2}\right)$$

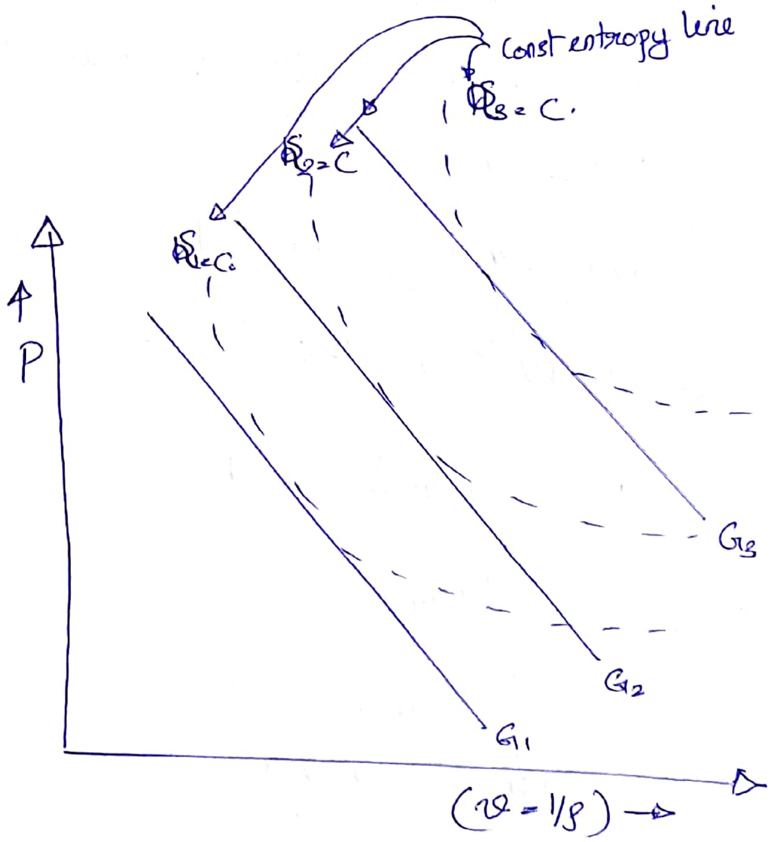
4) Equation of state

$$p = p(h, s)$$

$$\rho = \rho(h, s)$$

$$p = \rho R T$$

Rayleigh Line



$G_1 > G_2 > G_3$

$$PA + \rho A C^2 = \text{Const}$$

$$P + \rho C^2 = \text{Const}$$

$$\rho C = G$$

$$C^2 = \frac{G^2}{\rho^2}$$

$$P + \rho \frac{G^2}{\rho^2} = \text{Const}$$

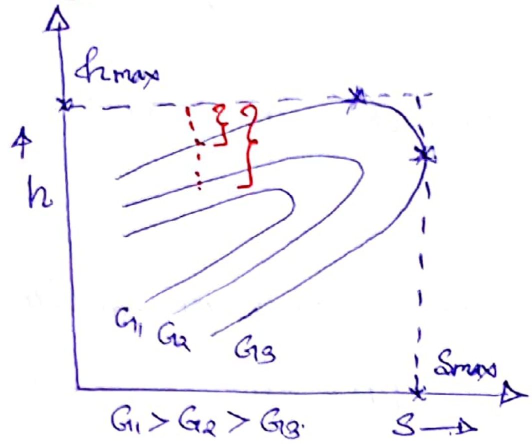
$$P + \frac{G^2}{\rho} = \text{Const}$$

$$P + G^2 v = \text{Const} \quad \text{—(I)}$$

A plot of P & v for different values of G gives curves representing Rayleigh ~~line~~ Flow called Rayleigh line & (I) is called Rayleigh line equation.

From equation of state $p = p(h, s)$

$\rho = \rho(h, s)$



$$p + \frac{G^2}{\rho} = \text{Const}$$

$$p(h, s) + \frac{G^2}{\rho(h,s)} = \text{Const}$$

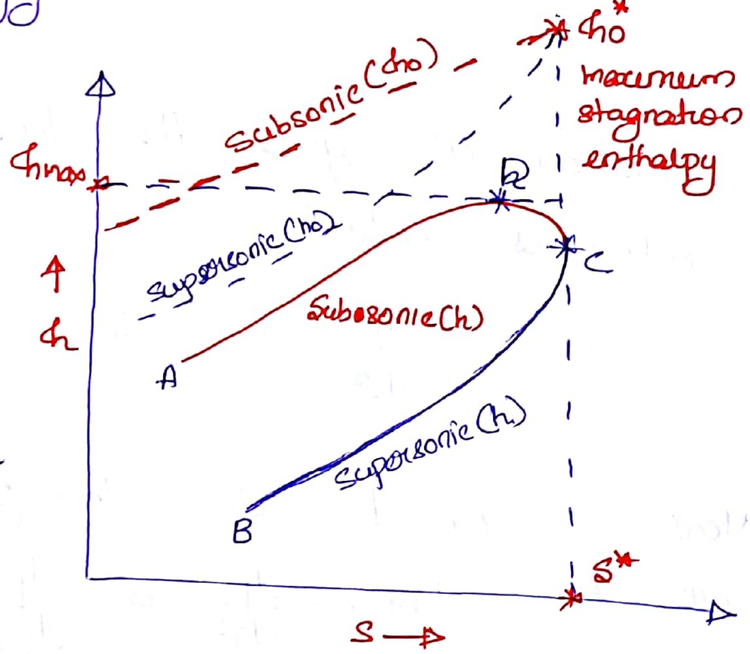
Using above equations we can plot Rayleigh flow with different G values in h-s diagram / Mollier diagram i.e.. Rayleigh line in Mollier Diagram.

when flow shifts from G_1 to G_3 mass flux reduces & Mach no. reduces

The shape of Rayleigh line is such that we get two significant points they are : Point of maximum enthalpy where $M = \frac{1}{\sqrt{\gamma}}$

Point of maximum entropy where $M = 1$

The point of maximum entropy divides the curve into two sections.

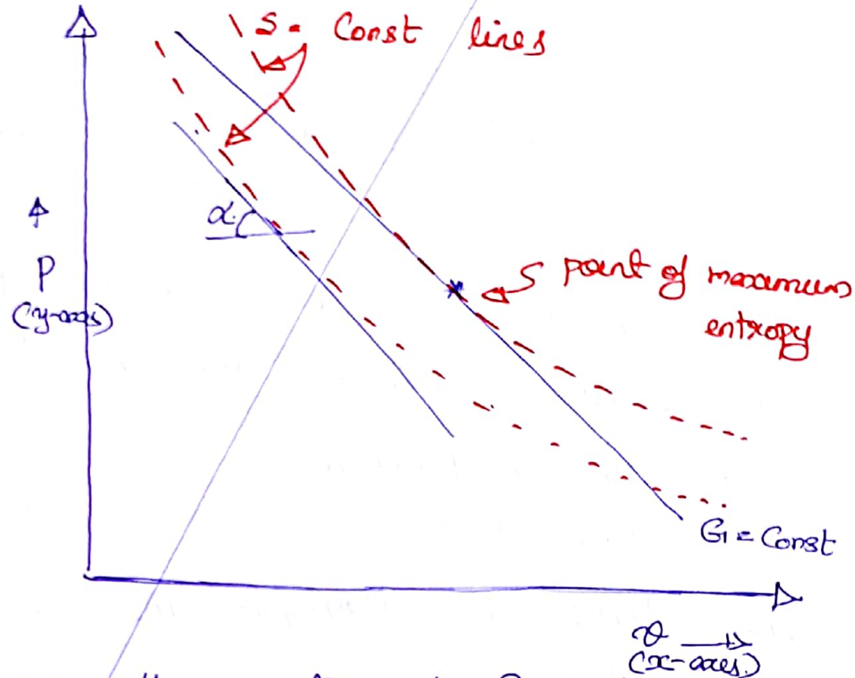


By comparing the K.E we can conclude that curve A-C represents subsonic Rayleigh flow.

& B-C represents supersonic Rayleigh flow.

Condition of Maximum Entropy in Raleigh (4) Flow

Consider a Rayleigh line of given value of G in p-v diagram



From momentum equation

$PA + \rho A C^2 = \text{Constant}$

$P + \rho C^2 = \text{Constant}$

$G = \rho C$

$\frac{G^2}{\rho^2} = C^2$

$P + \rho \frac{G^2}{\rho^2} = \text{Constant}$

$\boxed{P + G^2 v = \text{Constant}}$ gives the equation of Rayleigh line

Differentiating the above we get $dp + G^2 dv = 0$

$$-G^2 = \frac{dp}{dv} \quad \text{—— (1)}$$

Equation (1) gives the slope (dy/dx) of Rayleigh line ~~curve~~ which is constant

$$\frac{dp}{dv} = -G^2 = -\rho^2 C^2 = -\rho^2 a^2 M^2 \quad \text{—— (2)}$$

Now in general the slope of any curve in p-v diagram is given by

$$\tan\alpha = \frac{dp}{dv} = \frac{dp}{d(1/\rho)} = \frac{dp}{-\frac{1}{\rho^2}\,d\rho}$$

$\left\{ d(1/\rho) = d(\rho^{-1}) = -1 \times \rho^{-2}\, d\rho = -\frac{1}{\rho^2}\, d\rho \right.$

$$\frac{dp}{dv} = -\rho^2 \left(\frac{dp}{d\rho}\right)_s = -\rho^2 a^2$$

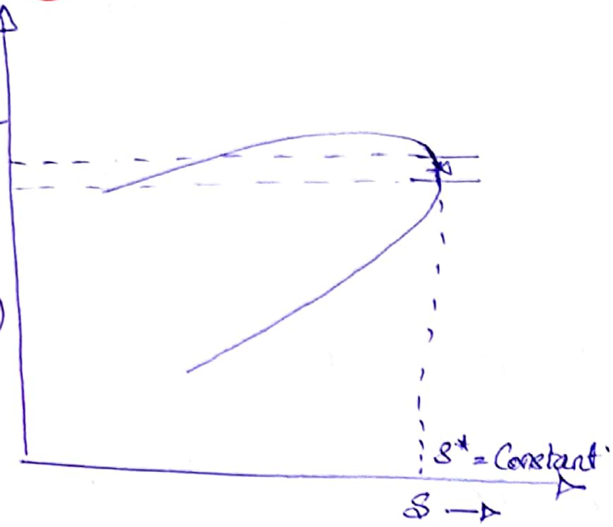
Const entropy line

$$\tan\alpha = -\rho^2 a^2 \quad \text{—— (3)}$$

Equation (2) & (3) are the same i.e. slope of curve in p-v coordinate

Condition of Maximum entropy

Consider an infinitesimal change along Rayleigh line in neighbour-hood of maximum entropy. We can approximate the process to be isentropic ($s^* = \text{Const}$)



along governing equations of Rayleigh line

$$G = \rho c = \text{Constant}$$

$$p + \rho c^2 = \text{Constant}$$

$$c^2 = \frac{G^2}{\rho^2}$$

$$p + \frac{G^2}{\rho} = \text{Constant} \quad \text{———} \quad (1)$$

differentiate (1) we get $dp + d\left(G^2 \rho^{-1}\right) = 0$

$$dp + -\rho^{-2} G^2 \, d\rho = 0$$

$$dp - \frac{G^2}{\rho^2} d\rho = 0$$

$$\frac{dp}{d\rho} = \frac{G^2}{\rho^2}$$

$$\frac{dp}{d\rho} = \frac{\rho^2 c^2}{\rho^2}$$

$$\frac{dp}{d\rho} = c^2$$

Since infinitesimal change is isentropic

$$\left(\frac{dp}{d\rho}\right)_s = c^2$$

$$a^2 = c^2$$

$$M^2 = 1$$

$$M = \pm 1$$

$$\underline{M = 1}$$

$$-\rho^2 a^2 M^2 = -\rho^2 a^2$$

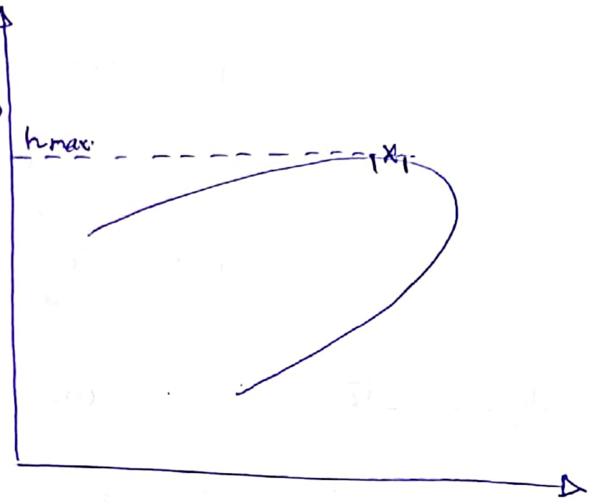
$$M^2 = 1$$

$$M = \pm 1$$

$$\Rightarrow M = 1.$$

Therefore condition for maximum entropy is Rayleigh line is $M = 1$.

Condition for Maximum Enthalpy



Consider an infinitesimal change along Rayleigh line in neighbourhood of maximum enthalpy. For this small change enthalpy remain the same i.e. h_{max}.

Therefore

$$dh = 0$$

$$dT = 0$$

From perfect gas equation

$$P = \rho R T$$

$$\ln P = \ln \rho + \ln R + \ln T$$

$$\frac{dp}{P} = \frac{d\rho}{\rho} + 0 + \frac{dT}{T}$$

Since its at point of maximum enthalpy $dT = 0$

$$\frac{dp}{P} = \frac{d\rho}{\rho}$$

$$\frac{dp}{d\rho} = \frac{P}{\rho} = RT \qquad \text{——— (1)}$$

From momentum equation

$$P + \rho c^2 = \text{Constant}$$

From Continuity equation $G = \rho c$

$$P + \frac{G^2}{\rho} = \text{Constant} \qquad \text{——— (2)}$$

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differentiating equation 2

$$dp + -\frac{C^2}{\rho^2} d\rho = 0$$

$$\frac{dp}{d\rho} = \frac{C^2}{\rho^2}$$

$$\frac{dp}{d\rho} = \frac{\rho^2 c^2}{\rho^2} = c^2 \quad \text{———} \quad (3)$$

Equating 1 & 2

$$c^2 = RT.$$

multiplying with γ. & dividing by γ on R.H.S.

$$c^2 = \frac{\gamma RT}{\gamma}.$$

$$c^2 = \frac{a^2}{\gamma}.$$

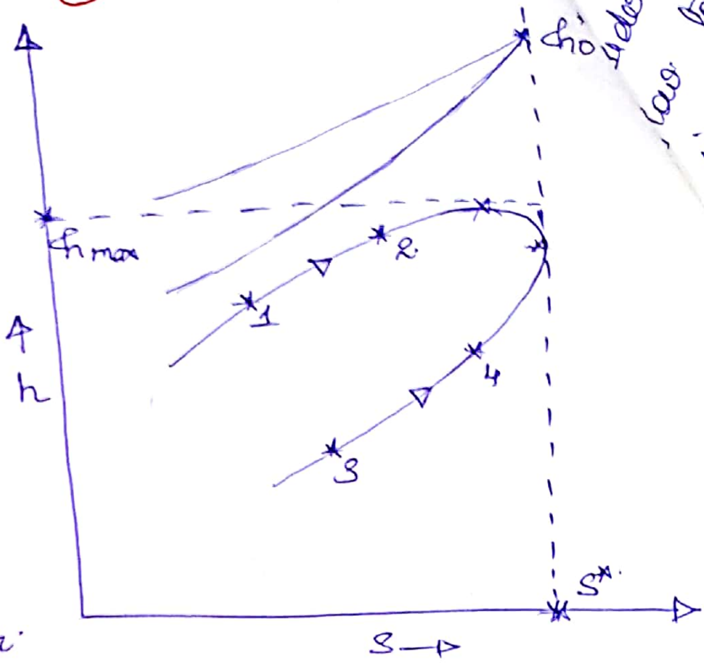
$$\frac{c^2}{a^2} = \frac{1}{\gamma}.$$

$$M^2 = \frac{1}{\gamma}$$

$$\boxed{M = \frac{1}{\sqrt{\gamma}}}$$

For $\gamma = 1.4$, M (choked) $= 0.84$

Effect of Heating & Cooling (7)



Consider Rayleigh flow with subsonic inlet condition. When heat is added, entropy should increase i.e. the flow moves towards right along subsonic Rayleigh line.

The pressure (static & stagnation) drops, density decreases.

Temperature (static & stagnation) increases and velocity of flow & Mach number increased.

HEATING RESULTS IN ACCELERATION OF SUBSONIC FLOW

This acceleration continues till point of maximum entropy (M=1) is reached. Flow donot go past in supersonic flow as it would be violation of 2nd Law of thermodynamics.

Consider Rayleigh flow with supersonic inlet condition. When heat is added entropy increased and moves to right along Rayleigh line.

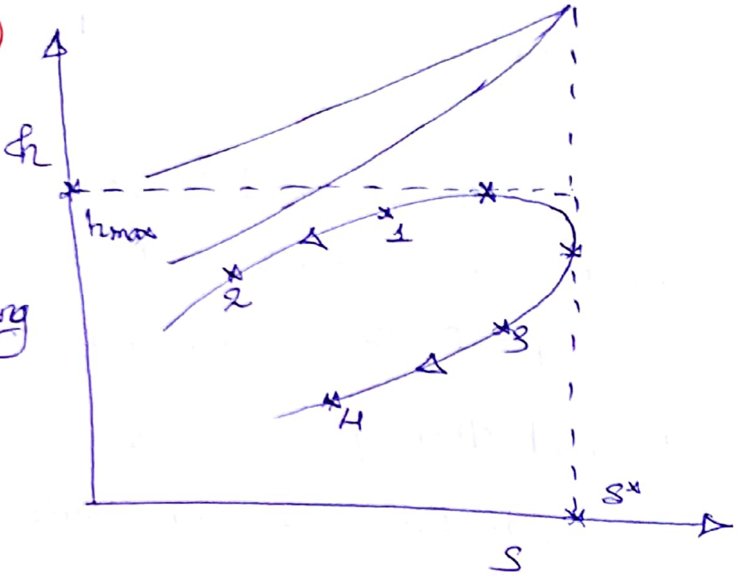
The static pressure, density, Temperature (both) increases

velocity, Mach number, stagnation pressure decreases.

HEATING RESULTS IN DECELERATION OF SUPERSONIC FLOW

This deceleration continues till point of maximum entropy is reached. Further heating causes normal shock to occur in flow making it subsonic.

Consider inlet conditions of Rayleigh flow to be subsonic. When fluid is cooled entropy decreases. (entropy of surrounding increases $\Rightarrow \Delta S_{surr} > 0$) causing the flow to move towards left of Rayleigh line.



Static pressure, density increased.
Temperatures (T & T_0) decreases, P_0, Mach number, velocity of fluid decreases

$$\boxed{\text{COOLING RESULTS DECCELERATION OF SUBSONIC FLOW.}}$$

This cooling can continue till velocity of fluid becomes zero

Consider inlet conditions of Rayleigh flow to be supersonic. As the heat is removed from fluid (cooled) entropy decreases and exit condition lies towards left on Rayleigh line

static pressure & ~~stagnation pressure~~, density decreases.
Temperatures (T & T_0) decreases.
velocity, Mach number & stagnation pressure increases.

$$\boxed{\text{COOLING RESULTS IN ACCELERATION OF SUPERSONIC FLOW}}$$

This cooling can continue till fluid velocity becomes maximum

REGION BETWEEN POINT OF h_{max} & S_{max} (9)

When heat is added to subsonic flow, it cause pressure & density to decrease and temperature to increase. This trend continues till point of maximum enthalpy is reached ($M = 1/\sqrt{\gamma}$).

Beyond this point when heat is added, the drop in density is such that a very high increase of kinetic energy is required

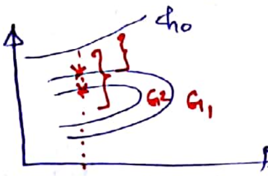
$$\left\{ \rho C = \text{Constant} \quad \text{when } \rho\downarrow \; C\uparrow \Rightarrow \tfrac{1}{2}C^2\uparrow \right\}$$

For this high K.E., apart from the heat added to flow, the enthalpy (heat energy) of fluid is also used. This cause static enthalpy & thereby static temperature to decrease even when heat is being added. This drop continues till point of maximum entropy is reached ($M = 1$)

$\boxed{\text{HENCE BETWEEN } M = \frac{1}{\sqrt{\gamma}} \text{ \& } M = 1\text{, HEATING OF FLUID SHOWS TEMPERATURE DROP i.e. COOLING EFFECT.}}$

While moving from $M = 1$ to $M = \frac{1}{\sqrt{\gamma}}$ i.e. cooling, the phenomena gets reversed and cooling results in increasing of temperature

Choking in Rayleigh Flow



$C_2 > C_1$
$G_1 < G_2$

When flow reaches $M = 1$ because of heat addition, the flow is said to be "THERMALLY CHOKED".

For subsonic flow heat addition beyond $M = 1$ cause upstream flow parameter to change, it reduces Mach number and reduces mass flux such that by the time complete heat is added flow exits with $M = 1$.

For supersonic flow heating beyond $M = 1$ causes normal shock to appear.

Heat Transferred during Rayleigh Flow (10)

For Rayleigh flow energy equation from SFEE

$$Q = \left(h_2 + \frac{C_2^2}{2}\right) - \left(h_1 + \frac{C_1^2}{2}\right)$$

$$Q = h_{02} - h_{01}$$

$$Q = C_p(T_{02} - T_{01})$$

$$Q = C_p T_{01}\left(\frac{T_{02}}{T_{01}} - 1\right)$$

$$Q = \frac{C_p T_{01}}{T_1} \times T_1 \left\{\frac{T_{02}}{T_{01}} - 1\right\}$$

$$\frac{Q}{C_p T_1} = \frac{T_{01}}{T_1}\left\{\frac{T_{02}}{T_{01}} - 1\right\} \quad \text{--- (1)}$$

$\frac{T_{01}}{T_1}$ from isentropic relation we have.

$$\frac{T_{01}}{T_1} = 1 + \frac{\gamma-1}{2} M_1^2 \quad \text{--- (2)}$$ { Eq 8.2, Page 4 gas tables

$$\frac{T_{02}}{T_{01}} = \frac{T_{02}/T_0^*}{T_{01}/T_0^*}$$

$$\frac{T_{02}}{T_0^*} = \frac{2(1+\gamma) M_2^2 \left(1 + \frac{\gamma-1}{2} M_2^2\right)}{1+\gamma M_2^2}$$

$$\frac{T_{01}}{T_0^*} = \frac{2(1+\gamma) M_1^2 \left(1 + \frac{\gamma-1}{2} M_1^2\right)}{1+\gamma M_1^2}$$

} Eq 8.4, Page 10 Gas tables

$$\frac{T_{02}}{T_{01}} = \frac{2(1+\gamma) M_2^2 \left(1 + \frac{\gamma-1}{2} M_2^2\right)}{1+\gamma M_2^2} \times \frac{1+\gamma M_1^2}{2(1+\gamma) M_1^2 \left(1 + \frac{\gamma-1}{2} M_1^2\right)} \quad \text{--- (3)}$$

Substituting (3) & (2) in (1) we get (11) after rearranging and simplification

$$\boxed{\frac{Q}{C_p T_1} = \left\{1+\frac{\gamma-1}{2}M_1^2\right\}\left\{\left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]^2\left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]\left[\frac{M_2}{M_1}\right]^2 - 1\right\}}$$

$\frac{Q}{C_p T}$ represents non dimensionalised quantity of heat transfer.

For maximum heat transfer $M_2 = 1$, $Q = Q_{max}$

$$\frac{Q_{max}}{C_p T_1} = \left\{\left(1+\frac{\gamma-1}{2}M_1^2\right)\right\}\left\{\left(\frac{1+\gamma M_1^2}{1+\gamma}\right)^2\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M_1^2}\right)\left(\frac{1}{M_1^2}\right)-1\right\}$$

$$\frac{Q_{max}}{C_p T_1} = \frac{2+(\gamma-1)M_1^2}{2}\left\{\frac{(1+\gamma M_1^2)^2(\cancel{\gamma+1})}{2(1+\gamma)^{\cancel{2}} M_1^2(2+(\gamma-1)M_1^2)} - 1\right\}$$

$$\frac{Q_{max}}{C_p T_1} = \left\{\frac{(1+\gamma M_1^2)^2(\gamma+1)}{2(1+\gamma)M_1^2} - \frac{2+(\gamma-1)M_1^2}{2}\right\}$$

Further rearranging and simplification gives

$$\frac{Q_{max}}{C_p T_1} = \frac{(M_1^2-1)^2}{2(\gamma+1)M_1^2}$$

or in general $\boxed{\frac{Q_{max}}{C_p T} = \frac{(M^2-1)^2}{2(\gamma+1)M^2}}$ {equation 8-8, Page 10, Gastables}